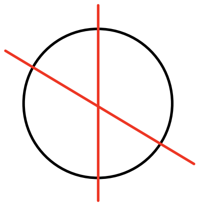
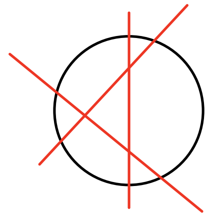
Cake cutting problem
What is the greatest number of pieces that a cake can be cut into with a given number of cuts? ($d=2$)



2 cuts
$P_2(2)=4$

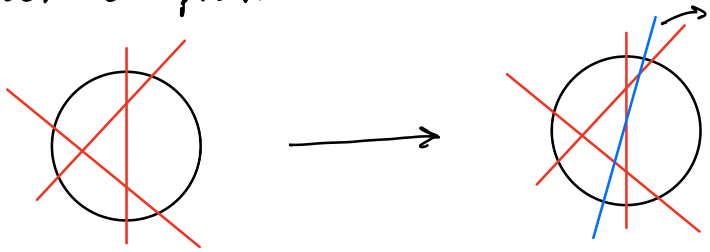


3 cuts
$P_2(3)=7$ . . .

Cuts N	1	2	3	4	5	. . .
Pieces $P_2(N)$	2	4	7	11	16	. . .

$$\begin{cases} 2 = 1+1 \\ 4 = 1+1+2 \\ 7 = 1+1+2+3 \\ \vdots \end{cases}$$

What it seems is that, having done $n-1$ "maximal" cuts, the number of pieces subdivided by the N-th cut equals <u>one more</u> than the number of previous cuts it meets



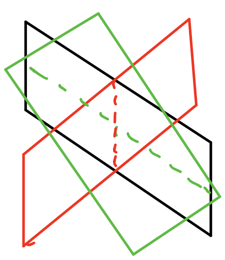
$\to +4$ regions, it meets 3 cuts

$P_2(N)=N+P_2(N-1)$
$P_2(1)=2$

We can give a close formula to $P_2(N)$:

$$P_2(N)=1+\sum_{i=1}^{N} = 1+\frac{N(N+1)}{2}$$

What about general d?
What we're asking is to compute the number of cells in an arrangement of n hyperplanes in d-dimensions in general position.
We can find a recursion relation like before, but now $d-1$ dimensional hyperplanes intersect and form $d-2$ dimensional objects, which were points in $d=2$.



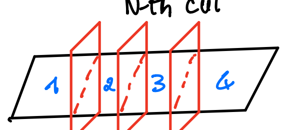
The N-th hyperplane will intersect $n-1$ previous ones and creates $P_{d-1}(N-1)$ new cells:

- $d=2$ $\quad P_1(N-1)$

$= N$

N-th cut

- $d=3$ $\quad P_2(N-1)$



$=\binom{N-1}{0}+\dots+\binom{N-1}{2}$

Check: $N=3 \Rightarrow \binom{2}{0}+\binom{2}{1}+\binom{2}{2}=1+2+1=\boxed{4}$

In essence, we have $\boxed{P_d(N) = P_d(N-1) + P_{d-1}(N-1)}$

Claim: $P_d(N) = \binom{N}{0} + \binom{N}{1} + \dots + \binom{N}{d}$

Proof: $\binom{N}{0} + \binom{N}{1} + \dots + \binom{N}{d-1} + \binom{N}{d} \overset{?}{=}$

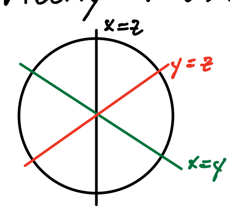
$$\overset{?}{=} \left.\binom{N-1}{0} + \binom{N-1}{1} + \dots + \binom{N-1}{d-1} + \binom{N-1}{d}\right\} P_d(N-1)$$

$$\left. + \binom{N-1}{0} + \binom{N-1}{1} + \dots + \binom{N-1}{d-1}\right\} P_{d-1}(N-1)$$

We just need to use $\binom{N-1}{k} + \binom{N-1}{k+1} = \binom{N}{k+1}$ $\square$

Symmetric cake-cutting

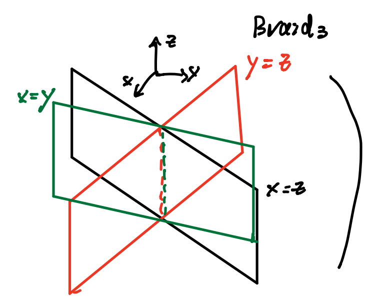
What are the possible ways to cut a perfectly round cake so that all pieces are congruent?



This can be reformulated as a problem of hyperplane arrangements, and in particular braid arrangement in the space of dimension

$\boxed{d = \text{number of cuts}}$

(i.e. you can see the figure as a $d=2$ projection of



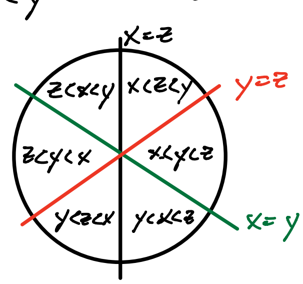
)

Each region of $\text{Braid}_{d=3}$ is on one side of the planes $x=y$, $x=z$, $y=z$. Therefore

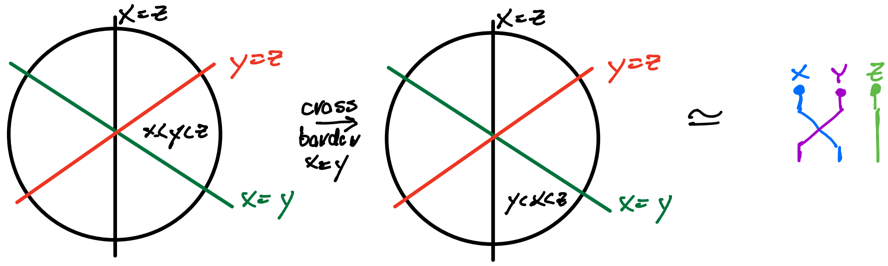
- either $x<y$ or $y<x$
- either $x<z$ or $z<x$
- either $y<z$ or $z<y$

$\Rightarrow$ 6 possibilities

$x<y<z$	$y<x<z$	$z<x<y$
$x<z<y$	$y<z<x$	$z<y<x$



Note: crossing one border means flipping one inequality



⇒ Braid group of permutations

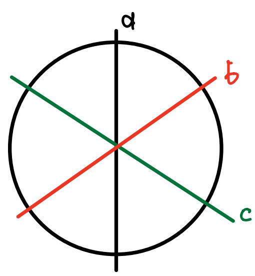
How many regions? $\square < \square < \square$

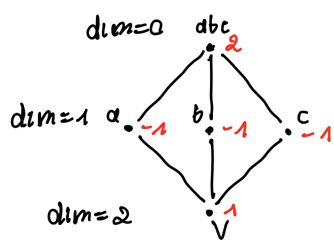
There are 3 possibilities for the first letter in the inequality, two for the second one and one for the last one

Sol. 6 $(=3!)$

This counting problems for arrangement has a nice formulation in terms of the characteristic polynomial. This will be a subject of the lecture next week, but since there will be no tutorial next week I thought it would be good to make a simple example here.

Let's take the intersection poset of the arrangement ordered by reverse inclusion





$V = \mathbb{R}^2$

Mobius function

The characteristic polynomial of an arrangement is

Def. (Stanley): $\chi_A(t) = \sum_{x \in L(A)} \mu(x)\, t^{\dim(x)}$

Let's apply it to our case:

$1 \cdot t^2 - 3 \cdot t + 2 = \chi_A(t) = t^2 - 3t + 2$

Theorem (Zaslavsky, 1975): $|\chi_A(-1)|$ = number of regions formed by the arrangement A in $\mathbb{R}^d$

$|\chi_A(-1)| = 6$ ✓

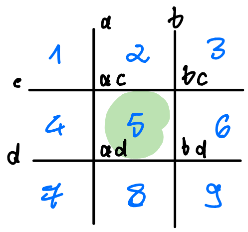
Even more

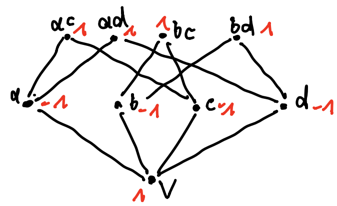
Theorem (Zaslavsky): $|\chi_A(+1)|$ = number of bounded regions formed by the arrangement A in $\mathbb{R}^d$

In the previous case

$|\chi_A(+1)| = 0$ ✓

Let's consider a non-trivial example





$\chi_A(t) = t^2 - 4t + 4$

$|\chi_A(+1)| = 1$ ✓

$|\chi_A(-1)| = \boxed{9}$ ✓