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Galactic magnetic fields

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Abstract. The plan of this work is to study the mechanism of generation of a seed magnetic field via second order perturbation theory, generated by the differential rotational velocity of ions and electrons

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1 Introduction of the problem

1.1 General description of galactic magnetic fields

Magnetic fields are ubiquitous in the Universe. They are present in our solar system, in stars, in the Milky Way, in other low and high redshift galaxies, in galaxy clusters, in superclusters and even in voids of the Large Scale Structure (LSS). The magnetic field strength in galaxies is always of the order of $1 \div 10\mu G$, independently of the galaxy redshift, and also the magnetic fields in clusters are of the order of μG [4]. These magnetic fields have large and extensive domains in which they are nearly uniform and of comparable strength, even thought they cannot be strongly homogeneous because of the uniformity of the CMB. The statements collected above rest on various detection techniques ranging from Faraday rotation, to synchrotron emission, to Zeeman splitting of clouds of molecules with an unpaired electron spin [3]. The experimental evidence suggests also that the strength of these magnetic fields is, in the first approximation, independent on the physical size of their domains. Magnetic fields in astronomical structures have infact different sizes, from stars ($R \simeq 10^6 km$) to galaxy clusters ($R \simeq 10^{19} km$), and this is very surprising. It is plausible to argue that large-scale magnetic fields have comparable strengths at large scales because the initial conditions for their evolutions (magnetogenesis) were the same, for instance at the time of the gravitational collapse of the protogalaxy.

The origin of the magnetic fields observed in the galaxies and in the clusters of galaxies is still unknown. Many people suggested that magnetic fields are produced mainly by amplification of pre-existing weaker magnetic fields via different type of dynamo (conversion of the kinetic energy of the turbulent motion of the conductive interstellar medium into magnetic energy) and via flux-conserving compression during gravitational collapse accompanying structure formation [2]. These magnetic fields has to live in a medium with high electrical conductivity in order to survive, and this condition is indeed fulfilled for the cosmic medium during most of the evolution of the Universe. Because of the Universe high conductivity, two important quantities are almost conserved during Universe evolution: the magnetic flux and the magnetic helicity. Regarding planets or stars (and partially galaxies), magnetic fields tipically dissipate their energy into thermal and turbolent motions of astrophysical plasmas on short distance scales, so a continuous re-generation of the field is needed on time scales shorter than the life time of the astronomical object carrying the field [4]. Weak magnetic fields on the largest distance scales, from the large scale fields in the galaxies to those in galaxy clusters, might not have enough time to dissipate their energy into plasma motions.

Anyway, today the efficiency of such a kind of MHD engines has been put in question both by improved theoretical work and new observations of magnetic fields in high redshift galaxies [2]. Moreover in any case the dynamo and compression amplification mechanisms can act only if a seed magnetic field is present (that could be tiny or not), which has to be generated by a different mechanism that preexist the structure formation epoch. Therefore the origin of galactic magnetic fields has to be traced back to a time comparable, at least, to that of galaxy formation. Magnetic fields may have partecipated to a relevant number of processes which took place in the early Universe, but unfortunatly the existing data on magnetic fields in galaxies or galaxy clusters can only provide indirect constraints on the origin and the properties of the seed fields.

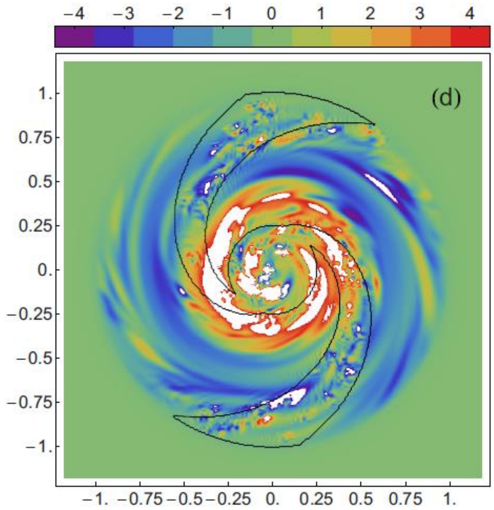

Figure 1: Typical azimuthal magnetic field component in a dynamo model, thanks to R. Stepanov, ICMM Perm

1.2 A brief report on dynamo and compression amplication mechanisms

In an electrical conducting fluid with electrical conductivity σ, the time evolution of the magnetic field $\vec{B}$ (magnetic diffusivity equation) is given by [3]

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4\pi\sigma} \tag{1.1}$$

that leads to a simple description of the conversion from kinetic energy to magnetic energy. The first time derivative of the magnetic field results from the balance of the dynamo term (first term on the RHS) that contains the bulk velocity of the plasma $\vec{v}$ and the magnetic diffusivity who damps the magnetic field (second term on the RHS). If the dynamo term dominates, the magnetic field could be amplified thanks to the differential rotation of the plasma. Generally speaking, also hydrodynamic turbolence and fast reconnection of magnetic field lines are required in order to provide an efficient dynamo mechanism. MHD can be studied in two different (but complementary) limits: the ideal (superconductive) limit ($\sigma \to +\infty$) and the real (resistive) limit (σ finite). This is related to two quantities, namely [2]

$$\frac{d}{dt}\int_{\Sigma} \vec{B} \cdot d\Sigma = -\frac{1}{4\pi\sigma}\int_{\Sigma} \nabla \times (\nabla \times \vec{B}) \cdot d\Sigma \qquad \text{magnetic flux} \tag{1.2}$$

$$\frac{d}{dt}\mathcal{H} = \frac{d}{dt}\int_V d^3x \vec{B} \cdot \vec{A} = -\frac{1}{4\pi\sigma}\int_V d^3x \vec{B} \cdot (\nabla \times \vec{B}) \qquad \text{magnetic helicity} \tag{1.3}$$

where Σ is an arbitrary closed surface that moves with the plasma, and V is a volume through the boundary of which no magnetic field lines cross. In the ideal MHD limit the magnetic flux, as well as magnetic helicity, is exacly conserved whereas in the resistive limit the magnetic flux and helicity are dissipated with a rate proportional to σ^{-1}. The conservation of magnetic flux implies that lines of force move together with the fluid, and the field in the ideal MHD limit is said to be frozen-in. Assuming that the Universe expands isotropically and disregarding any other effect that could produce a variation of the intensity or the direction of the magnetic field, if the field is frozen-in then it scales as (called also conformal scaling)

$$B(t) = B(t_i)\frac{a(t_i)^2}{a(t)^2} \tag{1.4}$$

Instead the conservation of magnetic helicity (closely analogous to vorticity) corresponds on the conservation of the topological properties of the magnetic flux lines. Infact $\mathcal{H}$ can be identified with the Chern-Simon number, and is proportional to the sum of the number of links and twists of the magnetic field lines. Defining then L as the typical scale of spatial variation of the magnetic field intensity, the typical time scale of resistive phenomena turns out to be $t_\sigma = 4\pi\sigma L^2$. Suppose that the time scale of the system is given by $t_U \simeq H_0^{-1} \simeq 10^{18}s$: then $L_\sigma = \sqrt{t_U \sigma^{-1}} \simeq 1\text{AU}$ gives an upper limit on the diffusion scale for a magnetic field whose lifetime is comparable with the age of the Universe at the present epoch. That means that magnetic fields with correlation scale $L < L_\sigma$ are diffused, and we are in the resistive regime: this is consistent with the experimental evidence that in galaxies there are no magnetic fields coherent over scales smaller than 10^{-5} pc [3]. In the following we present a simple argument in order to estimate the required strength of the primordial magnetic field. Since the gravitational collapse occurs at high conductivity, the magnetic flux and the magnetic helicity are both conserved. Right before the formation of the galaxy a patch of matter of roughly 1 Mpc collapses by gravitational instability. Right before the collapse the mean energy density of the patch, stored in matter, is of the order of the critical density of the Universe. Right after collapse the mean matter density of the protogalaxy is, approximately, six orders of magnitude larger than the critical density. Since the physical size of the patch decreases from 1 Mpc to 30 kpc the magnetic field increases (see equation 1.4) of a factor $\left(\frac{\rho_a}{\rho_b}\right)^{\frac{2}{3}} \simeq 10^4$ where ρ_a and ρ_b are respectively the energy densities right after and right before gravitational collapse. The correct initial condition in order to turn on the dynamo instability would be $B \sim 10^{-23}G$ over a scale of 1 Mpc, right before gravitational collapse. The estimates relies on the assumption that the amplification occurs over thirty e-folds while the magnetic flux is completely frozen in, but in the real situation the achievable amplification is much smaller and a good seed would be $B \geq 10^{-13}G$ before the collapse.

Regarding the other (but not unique) possible mechanism related to the generation of galactic magnetic fields, it deals with the protogalactic cloud collapse. The magnetic field results directly from a primordial field which gets adiabatically compressed in the collapse, and analogously to the equation 1.4 we can conclude that the conservation of magnetic flux in the intergalactic medium implies

$$B_{\text{prim}} = B_{\text{gal}}\left(\frac{\rho_{\text{cosmic}}}{\rho_{\text{gal}}}\right)^{\frac{2}{3}} \tag{1.5}$$

The present time ratio between the interstellar medium density in the galaxies and the density of the IGM is $\frac{\rho_{\text{IGM}}}{\rho_{\text{gal}}} \sim 10^{-6}$ and $B_{gal} \sim 10^{-6}G$, therefore the required strength of the cosmic magnetic field at the galaxy formation time ($z \sim 5$), adiabatically rescaled to present time, is $B_{prim} \sim 10^{-10}G$. This agree with with our observational limits and may produce observable effects on the anisotropies of the CMB background radiation.

Astrophysicists have to their disposal some particular informations: the observations of intensity and spatial distribution of the galactic and intergalactic magnetic fields and the structure of magnetic fields in objects at high redshift. Recent observations of strong magnetic fields in galaxy clusters suggest that the origin of these fields may indeed be primordial. It is useful to observe that primordial magnetic fields are not necessarily produced in the early Universe, i.e. before recombination time. Several alternative astrophysical mechanisms have been proposed like the generation of the fields by a Biermann battery effect. In the presence of a thermal plasma of protons and electrons, the generalized Ohmic electric field has a termoelectric correction $\vec{E}_{term} \propto -\frac{\nabla p_e}{en_e}$ where $p_e = (k_B n_e T_e)$ is the electron pressure n_e and T_e are the electron density and temperature. When the temperature and density gradients are misaligned, taking the curl of $\vec{E}_{term}$ leads to

$$\nabla \times \vec{E}_{term} \propto -\frac{\nabla n_e \times \nabla p_e}{en_e^2} \tag{1.6}$$

and therefore the integral of the electric field over a closed loop in the plasma is nonzero. According to Faraday's law, this electromotive force generates a magnetic flux. Biermann's battery can explain both a top down and bottom up scenario for magnetogenesis, but the generated magnetic fields are very weak ($B \sim 10^{-20} \div 10^{-10} G$) -especially for galaxy clusters- and they need to be amplified by some kind of plasma instabilities (magnetorotational) or by dynamo mechanism. Therefore such a scenario would lead us to face an unnatural situation where two different mechanisms are invoked for the generation of magnetic fields in galaxies and clusters, which have quite similar characteristics and (presumably) merge continuously at the border of the galactic halos. One of the other possibility is that magnetic fields may have been generated by batteries powered by starbursts or jet-lobe radio sources (AGNs, Active Galactive Nucleous). However, preexisting magnetic fields may be required to trigger starbursts or to carry away the huge angular moment of the in-falling matter of black holes: a primordial seed magnetic field is generally required [3]. In conclusion both the primordial and the astrophysical hypothesis for the origin of the seeds demand an efficient (large-scale) dynamo action.

2 A primordial seed for the magnetic field

2.1 Some remarks on primordial magnetic fields

We have seen in the previous paragraphs that there remain unresolved difficulties in explaining how the seed magnetic fields lead to fields of the observed strength and coherence scales. The possibility of the fields being completely primordial (created in the early universe, during inflation or during subsequent phase transitions) has to be taken into account, but unknown physics (non minimal coupling with EM field) must be invoked in order to explain the present galactic magnetic field at all scales. Infact the electroweak transition and the QCD transition are both not first order and actually they are not even true phase transition but just crossovers: this implies that magnetic fields could be produced only on very small scales, up to Hubble radius at magnetogenesis. However many modifications of the SM (Standard Model) lead to a first order (electroweak) phase transition, that proceeds via bubble nucleation which is a very violent event likely to lead to turbulence in the cosmic plasma. In a highly conducting cosmic plasma, turbulent flow generates eddies and magnetic fields in the plasma as a consequence of MHD turbulence [2].

However primordial magnetogenesis also takes place during recombination, as we can see using Maxwell theory and standard cosmological perturbation theory for the cosmic plasma. Within first order cosmological perturbation theory and within the strong coupling limit of electrons and protons, no magnetic fields form due to the inhomogeneities of the matter distribution of the Universe. For this to happen we need a current j with non-vanishing vorticity, since by Ampere's law in Gaussian units $\Delta\vec{B} = -\frac{4\pi}{c}\boldsymbol{\nabla}\times\vec{j}$. In the linear cosmological perturbation theory the charge current is $\vec{j} = e(n_p - n_e)\vec{v}$ where $\vec{v}$ is a scalar perturbation, i.e. a gradient. So even if at second order in the strong coupling limit in order to get $n_p \neq n_e$, we need to go to the second order in inhomogeneities in order to achieve $\boldsymbol{\nabla}\times\vec{j} = e\boldsymbol{\nabla}(n_p - n_e)\times\vec{v} \neq 0$.

2.2 General description of the setting

Let's consider a general spacetime with a Lorentzian metric $g_{\mu\nu}$ of signature $(-,+,+,+)$. We introduce a family of observers with worldlines tangent to the timelike 4-velocity vector $u^\alpha = \frac{dx^\alpha}{d\tau}$ where τ is the observers' proper time, so that $u^\alpha u_\alpha = -1$. The vector u^α determines the time direction, whereas the tensor $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ projects orthogonal to the 4-velocity u^α into observers' instantaneous rest space at each event. The Ehlers-Ellis 3+1 formalism is a covariant Lagrangian approach; i.e., every quantity has a natural interpretation in terms of observers comoving with the fundamental 4-velocity [6, 7].

The vector field u^α and its tensor counterpart $h_{\alpha\beta}$ allow for a unique decomposition of every spacetime quantity into its irreducible timelike and spacelike parts. These fields are also used to define the covariant time and spatial derivatives of any tensor field $T^{\gamma\delta\ldots}_{\alpha\beta\ldots}$ by $\dot{T}^{\gamma\delta\ldots}_{\alpha\beta\ldots} = u^\kappa \nabla_\kappa T^{\gamma\delta\ldots}_{\alpha\beta\ldots}$ and $D_\kappa T^{\gamma\delta\ldots}_{\alpha\beta\ldots} = h^\mu_\kappa h^\nu_\alpha h^\rho_\beta h^\gamma_\zeta h^\delta_\upsilon \cdots \nabla_\mu T^{\zeta\upsilon\ldots}_{\nu\rho\ldots}$.

We can decompose the projected covariant derivative $D_\nu u_\mu = h^\alpha_\mu h^\beta_\nu \nabla_\beta u_\alpha = \nabla_\nu u_\mu + u_\nu u^\gamma \nabla_\gamma u_\mu$ into its antisymmetric, symmetric traceless and trace part

$$\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} \tag{2.1}$$

with $\omega_{\mu\nu} = \frac{1}{2}(D_\mu u_\nu - D_\nu u_\mu)$, $\sigma_{\mu\nu} = \frac{1}{2}(D_\mu u_\nu + D_\nu u_\mu) - \frac{1}{3}\theta h_{\mu\nu}$ and $\theta = \nabla_\alpha u^\alpha$. These quantities are known as the rotation tensor, shear tensor and expansion of the congruence family of geodesics defined by u^α. By construction we have $\sigma_{\mu\nu}u^\mu = 0 = \omega_{\mu\nu}u^\mu = A_\mu u^\mu$, where the 4-projected acceleration vector A_μ is equal to $u^\gamma\nabla_\gamma u_\mu$. We note that $D_\nu u_\mu$ describes the relative motion of neighbouring observers: the volume scalar determines the average separation between neighbouring observers, the effect of the vorticity is to change the orientation of a given fluid element without modifying its volume or shape whereas the shear changes the shape but leaves the volume unaffected. If we define the effective volume element in the observer's instantaneous rest space as $\epsilon_{\mu\nu\lambda} = u^\tau\epsilon_{\tau\mu\nu\lambda}$, where $\epsilon_{\tau\mu\nu\lambda}$ is the spacetime volume (four dimensional totally antisymmetric tensor with $\epsilon_{0123} = \sqrt{-g}$), then the vorticity vector will be defined as $\omega_\nu = \epsilon_{\mu\nu\lambda}\omega^{\nu\lambda}$ with $\omega^{\nu\lambda}$ projected antisymmetric vorticity (rotation tensor).

In the following, we try to follow mainly the notation of the paper of Fenu et al. [13] but using other references as a primary source in order to explain clearly almost all details of the calculations. Let's consider the stress-energy tensor of a species s $T_s^{\mu\nu}$, where s can be any type of matter, including electromagnetic fields and scalar fields. In our specific case we deal with r = p, e, γ, F (protons, electrons, photons and EM field respectively). The 4-velocity of species s is $u_s^\mu = \gamma_s(u^\mu + v_s^\mu)$, with $u_\mu v_s^\mu = 0$ and $\gamma_s = (1 - v_s^2)^{-\frac{1}{2}}$. The global continuity equation ensures that $\sum_s \nabla_\nu T_s^{\mu\nu} = 0$, whereas for every single species we have $\nabla_\nu T_s^{\mu\nu} = \sum_r C_{sr}^\mu$, where $C_{sr}^\mu = -C_{rs}^\mu$ encodes all the effects of interactions with species r. With respect to our observers, the energy-momentum tensor of a general fluid decomposes into its irreducible parts as [7]

$$T_s^{\mu\nu} = \rho_s u^\mu u^\nu + p_s h^{\mu\nu} + 2q_s^{(\mu}u^{\nu)} + \pi_s^{\mu\nu} \tag{2.2}$$

where there is an energy density $\rho_s = m_s n_s$, a pressure $p_s = \frac{1}{3}v_s^2\rho_s$, a momentum density $q_s^\mu = \rho_s v_s^\mu$ and an anisotropic stress $\pi_s^{\mu\nu} = \rho_s\left(v_s^\mu v_s^\nu - \frac{1}{3}v_s^2 h^{\mu\nu}\right)$. In the rest frame, electron (s = e) and protons (s = p) are well described by dust matter, whereas photons (s = γ) are relativistic massless particles. Therefore there is a unique hydrodynamic 4-velocity, relative to which q_s^μ and $\pi_s^{\mu\nu}$ are identically zero and the effective pressure reduces to the equilibrium one. In formulas, there is a frame R in which $T_s^{R,\mu\nu} = (p_s^R + \rho_s^R)u_s^\mu u_s^\nu + p_s^R g^{\mu\nu}$, with $p_s^R = w_s\rho_s^R$ and $\rho_s^R = \frac{\rho_s}{\gamma_s^2}$. We choose consistently $w_p = w_e = 0$ (pressureless perfect fluid) and $w_\gamma = \frac{1}{3}$ (relativistic perfect fluid). Regarding electromagnetic field, Maxwell equations in general spacetime can be rewritten -working in Gaussian units- as $\nabla_{[\mu}F_{\nu\sigma]} = 0$ and $\nabla_\nu F^{\mu\nu} = j^\mu$. where $F^{\mu\nu}$ is the antisymmetric electromagnetic Faraday tensor and j^μ is the 4-dimensional electromagnetic current. The electric and magnetic field experienced by our u^α observer are $E_\mu = F_{\mu\nu}u^\nu$ and $B_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma}F^{\nu\sigma}$ (where $E_\mu u^\mu = B_\mu u^\mu = 0$). Following the decomposition of the 4-vector current with respect to the u^α congruence $j^\mu = \varrho u^\mu + j^{\perp,\mu}$, with $\varrho = -j^\alpha u_\alpha$ representing the charge density and $j^{\perp,\mu} = h^\mu_\nu j^\nu$ the projected current. Relative to a fundamental observer, each one of Maxwell's equations decomposes into a timelike and a spacelike component [7, 13]:

- Timelike component of Maxwell equations:

$$\dot{B}^\perp_\mu = -\frac{2}{3}\theta B_\mu + (\sigma_{\mu\nu} - \omega_{\mu\nu})B^\nu - \epsilon^{\mu\nu\rho}D_\nu E_\rho - \epsilon_{\mu\nu\lambda}a^\nu E^\lambda \tag{2.3}$$

$$\dot{E}^\perp_\mu = -\frac{2}{3}\theta E_\mu + (\sigma_{\mu\nu} - \omega_{\mu\nu})E^\nu - \epsilon^{\mu\nu\rho}D_\nu B_\rho - \epsilon_{\mu\nu\lambda}a^\nu B^\lambda - j^\perp_\mu \tag{2.4}$$

- Spacelike component of Maxwell equations (constraints):

$$D_\mu B^\mu = -\omega_\mu E^\mu \tag{2.5}$$

$$D_\mu E^\mu = \omega_\mu B^\mu + \varrho \tag{2.6}$$

In our specific case, if n_s is the number density in the rest frame for s = p,e and e is the unit charge, then $\varrho = e(\gamma_p n_p - \gamma_e n_e)$ and $j^{\perp,\mu} = e(\gamma_p n_p v_p^\mu - \gamma_e n_e v_e^\mu)$. Moreover in our specific case we have an electromagnetic field contribution to the energy-momentum tensor of the baryons

$$\left(\nabla_\nu T_B^{\mu\nu}\right)_{EM} = -\left(\nabla_\nu T_F^{\mu\nu}\right) = -\left(-\sum_B F^\mu_\sigma j^\sigma_B\right) = F^\mu_\sigma (q_B n_B u^\sigma_B) \tag{2.7}$$

where q_B is the baryon charge.

2.3 Euler equations for electrons and protons

The formalism described above applies for any covariant choice of u^α, but in a multi-fluid situation there could be different choices due to the distinct 4-velocities of different particles species. Each choice leads a different 3+1 covariant description, and one can regard such choice as partial (physical) gauge fixing. During the radiation era, the components of the multifluid are electrons, protons and photons and we can safely neglect the role of dark matter component (CDM). At temperatures $T \sim m_e$, where m_e is the mass of the electron, the couplings between electrons and protons (and generically ions) and electrons and photons are very strong. As we said before, there couldn't be any kind of differential motion between different particles species and no magnetic field is generated at that time. Afterwards, from the end of particle/anti-particle annihilation up to now ($T \lesssim m_e$) electrons and photons are still tightly coupled through Thomson scattering (the scattering between photons and protons is weaker by a factor of $(\frac{m_e}{m_p})^2$)[1] whereas protons and electrons are coupled by means of Coulomb scattering [10]. The difference between momentum transfer rates contributions of these interactions gives rise to a small difference in the electron and proton fluid rotational velocities, that bring to the generation of a current with nonvanishing vorticity (and therefore of a magnetic field). In order to write the Euler equation for the proton and electron velocities, let's recap that $\nabla_\nu T_s^{\mu\nu} = \sum_r C^\mu_{sr}$ for every species s, where C^μ_{sr} is evaluated in the u^α frame. Any difference between such u^α's will be $O(\epsilon)$ in the almost-FLRW case, that is the case in which quantities vanish in the FLRW limit[2], because it is the only possibility to linearize about a FLRW model in a consistent 1+3 covariant and gauge invariant way. Having chosen a preferred u^α-frame, we need to consider the relative velocities v_s^μ of each species in that frame. If they are close to each other (all species have nonrelativistic motion with respect to the u^α-frame), then $O(v_s^2)$ terms can be dropped, that is we can linearize in relative velocities even without linearizing in kinematic, dynamic and elecrodynamics quantities [6]. This is what we have to do in order to find all contributions due to the Coulomb and Thomson interactions [11], whereas for the coupling of protons and electrons with the electromagnetic field we already found the exact expression in equation 2.7. The idea is to use relativistic kinetic theory in order to build up a full microscopic description of the evolution of the phase space distribution $f_s(x,p)$ for one species s by means of the relativistic Boltzmann equation

[1] After electron-positron annihilation, the average photon energy is much less than the electron rest mass and the electron thermal energy may be neglected, so that the Compton interaction between photons and electrons, the dominant interaction between radiation and matter, may reasonably be described in the Thomson limit

[2] A FRW universe with a unique 4-velocity u^α is characterized by the conditions [6]:

- Dynamics: $D_a\rho = D_a p = 0, \quad q_a = 0, \quad \pi_{ab} = 0$
- Dynamics: $D_a\theta = 0, \quad A_a = \omega_a = 0, \quad \sigma_{ab} = 0$
- Gravito-electic and gravito-magnetic fields: $E_{ab} = H_{ab} = 0$

$\frac{df_s}{d\lambda} = p^\mu \frac{\partial f_s}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f_s}{\partial p^\mu} = C[f]$ where λ is a parameter, $p^\mu = \frac{dx^\mu}{d\lambda}$ is the momentum of the particle and $C[f]$ is the collision term. Regarding Coulomb interaction, using the Coulomb cross section σ_{Coul} and remembering the form of the collision term (proportional to the density n_B) it is easy to find the Coulomb contribution to the proton and electron momentum transfer rates, at first order in their velocities v^μ_B. Let's consider Thomson scattering instead: when the radiation is close to blackbody we do not require the full spectral behavior of the distribution multipoles, but only the energy-integrated multipoles. Moreover the 3+1 formalism forces us to use 3+1 covariant harmonics, that is we split the photon 4-momentum p^μ_γ into $p^\mu_\gamma = E_\gamma(u^\mu + e^\mu)$, where $e^\mu = \frac{p^{\mu,\perp}_\gamma}{E_\gamma}$ is the photon's spatial propagation direction, such that $e^\mu e_\mu = 1$ and $e^\mu u_\mu = 0$ and we expand [9]

$$f(x,p) = f(x,E,e) = F + F_\alpha e^\alpha + F_{\alpha\beta} e^\alpha e^\beta + ... = \sum_{l\geq 0} F_{A_l}(x,E) e^{\langle A_l \rangle} \tag{2.8}$$

$$C[f](x,E,e) = b + b_\alpha e^\alpha + b_{\alpha\beta} e^\alpha e^\beta + ... = \sum_{l\geq 0} b_{A_l}(x,E) e^{\langle A_l \rangle} \tag{2.9}$$

where $e^{A_l} = e^{\alpha_1} e^{\alpha_2} ... e^{\alpha_l}$, $e^{\langle A_l \rangle}$ is the symmetric trace-free part of e^{A_l} and provides a representation of the rotation group. $F_{A_l}(x,E)$ are the 3+1 covariant distribution function anisotropy multipoles that are irreducible since they are Projected, Symmetric and Trace-Free (PSTF), i.e.

$$F_{ab...z} = F_{\langle ab...z \rangle} \Leftrightarrow F_{ab...z} = F_{(ab...z)}, F_{ab...z} u^a = 0, F_{ab...z} h^{ab} = 0 \Rightarrow F_{A_l} = F_{\langle A_l \rangle} \tag{2.10}$$

whereas $b_{A_l}(x,E) = b_{\langle A_l \rangle}(x,E)$, called scattering multipoles encode the covariant aspects of particle interactions. The Boltzmann's equation is then equivalent to an infinite hierarchy of 3+1 covariant multipole equations for each photon energy E

$$L_{A_l}(x,E) = b_{A_l}\left[F_{A_m}(x,E)\right] \tag{2.11}$$

where $L_{A_l}(x,E)$ are the anisotropy multipoles of $\frac{df}{d\lambda}$.

The calculations are then pretty straightforward and following Fenu et al. [13] (having neglected all polarization effects, and also making the approximations mentioned above about the -nonrelativistic- velocity of the baryonic frame relative to the u^α frame) we can easily obtain also the Thomson contribution to the transfer momentum rate of the baryons. In conclusion we have (after the projection onto the observers, and indexing each baryon with B = e,p)

$$(\nabla_\nu T^{\mu\nu}_B)^\perp_{EM} = q_B n_B (E^\mu + \epsilon_{\mu\delta\tau} v^\delta_B B^\tau) \tag{2.12}$$

$$(\nabla_\nu T^{\mu\nu}_B)^\perp_{Thomson} = -n_e \left(\frac{m_e}{m_B}\right)^2 \rho_\gamma \sigma_T \left[\frac{4}{3}(\gamma_B v^\mu_B - \gamma_\gamma v^\mu_\gamma) + \frac{8}{15}\Theta^\mu_\delta v^\delta_B\right] + O(\epsilon v^2_B, v^3_B) \tag{2.13}$$

$$(\nabla_\nu T^{\mu\nu}_p)^\perp_{Coulomb} = -(\nabla_\nu T^{\mu\nu,\perp}_e)_{Coulomb} = e^2 n_e n_p \eta_C (\gamma_e v^\mu_e - \gamma_p v^\mu_p) + O(\epsilon v^2_B, v^3_B) \tag{2.14}$$

where q_B is the charge of the baryon, $\sigma_T = \frac{8\pi\alpha^2}{3m_e^2}$ and $\eta_C = \frac{\pi e^2 \sqrt{m_e}\log(\Lambda)}{T^{\frac{3}{2}}}$ with Λ as the Coulomb logarithm. Having extracted the Euler equation from the previous equations, we obtain finally (see the equation 44 of [6], after putting $w = c_s = 0$, multiplying by ρ_s and projecting)

$$m_s n_s (\dot{v}^{\mu,\perp}_s + A^\mu_s + K^\mu_s) = (\nabla_\nu T^{\mu\nu}_s)^\perp_{EM} + (\nabla_\nu T^{\mu\nu}_B)^\perp_{Thomson} + (\nabla_\nu T^{\mu\nu}_s)^\perp_{Coulomb} \tag{2.15}$$

$$K^\mu_s = \left(\frac{\dot{n}_s}{n_s} + \frac{4}{3}\theta + A_\nu v^\nu_s + \frac{1}{n_s} v^\nu_s D_\nu n_s + D_\nu v^\nu_s\right) v^\mu_s + (\sigma^{\mu\nu} - \omega^{\mu\nu}) v_{s,\nu} + v^\nu_s D_\nu v^\mu_s \tag{2.16}$$

where $r, s = p, e$.

2.4 Generation of the electric field

Since we know a priori that the final result will be independent from the difference $n_e - n_p$ (this is due to some approximations we will do, that are consistent with our situation), we require charge neutrality from the beginning $n_p = n_e = n$. The dynamical variable of the problem, as we said also before, is the velocity difference $\Delta v^\mu_{pe} = v^\mu_p - v^\mu_e$ and taking the sum of equation 2.15 for $s = e$ with $\beta = \frac{m_e}{m_p}$ times the same equation 2.15 for $s = p$ we obtain

$$m_e n(\Delta\dot{v}^{\mu,\perp}_{pe} + \Delta K^\mu_{pe}) = neE^\mu(1+\beta) + \beta\left[(\nabla_\nu T^{\mu\nu}_p)^\perp_{Coulomb} + (\nabla_\nu T^{\mu\nu}_p)^\perp_{Thomson}\right] - \tag{2.17}$$

$$- \left[(\nabla_\nu T^{\mu\nu}_e)^\perp_{Coulomb} + (\nabla_\nu T^{\mu\nu}_e)^\perp_{Thomson}\right] \tag{2.18}$$

where we neglected Lorentz force because it is of second order in velocities. Since by thermal collision the velocity of hydrogen atoms is close to the velocities of electrons and protons, we choose to work with the center of mass velocity of baryons defined by $v^\mu_B = \frac{\sum_s m_s v^\mu_s}{\sum_s m_s}$. Making all explicit in equation 2.17 and substituting each v_e and v_p we obtain

$$m_e n(\Delta\dot{v}^{\mu,\perp}_{pe} + \Delta K^\mu_{pe}) = neE^\mu(1+\beta) + (\beta+1)\left[e^2 n^2 \eta_C \Delta v^\mu_{pe}\right] - \tag{2.19}$$

$$-\frac{4}{3}\sigma_T n\rho_\gamma\left[(1-\beta^3)\left(\Delta v^\mu_{B\gamma} + \frac{2}{5}\Theta^{\mu\nu}v_{B,\nu}\right) - \frac{1+\beta^4}{1+\beta}\left(\Delta v^\mu_{pe} + \frac{2}{5}\Theta^{\mu\nu}\Delta v_{pe,\nu}\right)\right] \tag{2.20}$$

Now it is clear an electric field (and thus a magnetic field, via Maxwell equations 2.3, provided E has a transverse component) can be generated by a nonzero velocity difference $\Delta v_{pe,\nu}$ and $\Delta v_{B\gamma,\nu}$. Using Maxwell equations in the covariant 3+1 form and neglecting higher order terms, we can easily derive an expression for Δv^μ_{pe}

$$j^\mu_{pe} = en\Delta v^\mu_{pe} = \text{curl}B^\mu - \dot{E}^{\mu,\perp} - \frac{2}{3}\theta E^\mu + \sigma^{\mu\nu}E_\nu \tag{2.21}$$

To proceed further, we have to make some estimates about different time-scales of our problem, namely the time-scale of the evolution of the plasma τ_{evol} and of the Coulomb and Thomson interactions τ_C and τ_T. Skipping some technical details (the interested reader is referred directly to the Fenu's et al. article [13]), it follows that the largest contribution to the electric field is due to the velocity difference $\Delta v^\mu_{B\gamma}$ because other contributions are negligible in the period (from very high z to nowadays) and on the scale of interest. This is true even taking into account the recombination at $z \simeq 1080$, and we obtain the following expression for the electric field [12]

$$E^\mu = -\frac{1-\beta^3}{1+\beta}\frac{4\rho_\gamma\sigma_T}{3e}\left(\Delta v^\mu_{B\gamma} + \frac{2}{5}\Theta^{\mu\nu}v_{B,\nu}\right) \tag{2.22}$$

where the first contribution can be considered as an Ohm-like contribution (since it is proportional to the current density $e\Delta v^\mu_{B\gamma}$) and the second one is due to the anisotropic stress. It is important to notice that the contribution vanish for $\beta \to 1$ (due to the fact that it is originated from the velocity difference between protons and electrons in their interactions with photons) and that it is independent from the density of free electrons. The last remark ensures that the electric field is generated also after the recombination. A last word about the relation between $\Delta v^\mu_{B\gamma}$ and Δv^μ_{pe}: it can proved (throughout an estimation via time scales, as before) that $\Delta v^\mu_{pe} \ll \Delta v^\mu_{B\gamma}$ for all relevant times. This means that $\Delta v^\mu_{B\gamma} \to 0$ can be taken early in the calculations, but only provided that $\Delta v^\mu_{pe} \ll \Delta v^\mu_{B\gamma}$ (we can consider baryons as a single fluid).

In order to go further, we need to know how to deal with the explicit pertubations around Friedmann background. In the Poisson gauge, up to second order (i.e. no tensor perturbations), the FLRW metric can be written as (with conformal time η)

$$ds^2 = a^2 \left[-(1+2\Phi)d\eta^2 + 2B_i dx^i d\eta + (1-2\Psi)\delta_{ij}dx^i dx^j\right] \tag{2.23}$$

where Φ, Ψ are scalar perturbations and B_i is a vector divergenceless perturbation. It is useful to decompose perturbed quantities in a background orthonormal tetrad, i.e. in a local inertial frame: this facilitates the separation between the magnitude of the momentum u^α and its direction (represented by a spacelike unit vector). We will use the Latin alphabet for tetrad basis, that are defined by

$$e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab} \qquad e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab} \tag{2.24}$$

where $\eta = diag(-1,1,1,1)$ is the Minkowski metric. Since the tetrad labels run from 0 to 3, we use Latin indices starting from the letter a with values ranging from 1 to 3 to label the spacelike vectors or forms, whereas the label o is for the timelike vector and form in a tetrad. We work with a comoving tetrad related to the u^α, defined in such a way that $e^\mu_o = u^\mu$ holds. The gauge freedom related to the arbitrariness in the choice of a gauge for the perturbed metric can be encoded in a gauge a vector field X, defined as a vector field on the 5-dimensional manifold $\mathcal{N} = [0,1] \times \mathcal{M}$ with the property $X^4 = 1$, where $\mathcal{M}$ is the spacetime manifold [5]. If $\lambda \in [0,1]$, then $\mathcal{M}_\lambda = \mathcal{M} \times \{\lambda\}$ is equal to the FLRW universe for $\lambda = 0$ and to the perturbed FLRW universe for $\lambda = 1$. The integral curves of the vector field X define a one parameter group of diffeomorphisms $\phi_\lambda(\cdot)$ and they are always transverse to the spacetime leaves $\mathcal{M}_\lambda$, since the last component of X is always nonzero. Therefore we can construct the perturbed tetrads by pulling back tetrads of FLRW universe to the perturbed one, obtaining explicitly (remember that we are working with a comoving tetrad)

$$e^\mu_o = \frac{1}{a}(1 - \Phi + \frac{3}{2}\Phi^2)\delta^\mu_o - \frac{1}{a}B^a\delta^\mu_a \tag{2.25}$$

$$e^\mu_a = \frac{1}{a}(1 + \Psi + \frac{3}{2}\Psi^2)\delta^\mu_a \tag{2.26}$$

and the corresponding expressions for e^o_μ and e^a_μ. Derivatives along the tetrad vectors are defined as $\partial_a = e^\mu_a \partial_\mu$. We can also derive the (perturbed) Ricci rotation coefficients $\Omega_{abc} = \eta_{bd} e^d_\nu e^\mu_a \nabla_\mu e^\nu_c$ for our orthonormal tetrads in a straightforward way in order to make explicit the covariant derivative $\nabla_a X^c_b = e^\mu_a \partial_\mu X^c_b + \Omega_a{}^d{}_b X^c_d - \Omega_a{}^c{}_d X^d_b$.

2.5 Generation of the magnetic field

Let's restart from the Maxwell's equations 2.3. Rewriting each term on the RHS in the tetrad basis we obtain $(curl E)^a = \epsilon^{abc}\nabla_b E_c = \epsilon^{abc}\partial_d\left[(1-\Psi)E_c\right]$, $e^a_\mu \epsilon^{\mu\nu\rho}\dot{u}_\nu E_\rho = \epsilon^{abc}\dot{u}_b E_c = \epsilon^{abc}(\partial_b\Phi)E_c$. On the LHS we get $a^{-1}\partial_o(a^2 B^a)$, and remembering that at first order the electric field is curl-free ($\epsilon^{abc}\partial_b E_c = 0$) the evolution equation for the magnetic field in coordinate basis becomes

$$a^{-1}(a^2 B^a)' = -a\epsilon^{abc}\partial_b\left[(1+\Phi-\Psi)E_c\right] \tag{2.27}$$

Let us now consider the equation of the generated electric field 2.22: rewriting it using the ionization fraction $x_e = \frac{n_e}{n_e+n_H}$, neglecting $\beta \ll 1$ and recognizing that the RHS side is proportional to $(\nabla_\nu T_B^{\mu\nu})^{\perp}_{Thomson}$ we obtain $e(n_e + n_H)x_e E^\mu = (\nabla_\nu T_B^{\mu\nu})^{\perp}_{Thomson}$. Substituting the expression for E^μ into the equation 2.27 we get finally

$$(a^2 B^a)' = -\frac{a^2}{e(n_e + n_H)x_e}\epsilon^{abc}\partial_b \left[(1 + \Phi - \Psi)(\nabla^\nu T_{B,\nu c})^{\perp}_{Thomson}\right] \tag{2.28}$$

where B here is related to protons, electron and even hydrogen atoms.

3 Power spectrum and strength of the generated magnetic field

3.1 A second order equation for magnetogenesis with all sources contributions

It is more convenient from the numerical point of view to describe the angular dependence of the radiation functions using the normal modes component, that is to decompose the quantities evaluated in the local inertial frame into multipoles: [5]

$$\Theta_{a_1...a_l}(x^0,\vec{x})n^{a_1}...n^{a_l} = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sum_{lm} \Theta_l^m(\eta,\vec{k}) G_{lm}(\vec{k},x^0,\vec{x},\vec{n}) \tag{3.1}$$

$$G_{lm}(\vec{k},x^0,\vec{x},\vec{n}) = i^{-l}\left(\frac{4\pi}{2l+1}\right) e^{i\vec{k}\cdot\vec{x}} Y^{lm}(\vec{n}) \tag{3.2}$$

where the components $\Theta_l^m(\eta,\vec{k})$ are the time-dependent components in Fourier space of the spherical harmonics decomposition. The helicity basis is well suitable for the decomposition of tensors: we define

$$\bar{e}^i_{(0)} = -\bar{e}^i_3 \qquad \bar{e}^i_{(+)} = \frac{1}{2}(\bar{e}^i_1 + i\bar{e}^i_2) \qquad \bar{e}^i_{(-)} = -\frac{1}{2}(\bar{e}^i_1 - i\bar{e}^i_2) \tag{3.3}$$

and their inverse, keeping in mind that this 'vector' basis transforms in the background spacetime with the Euclidean metric δ^{ij}. It is particularly useful to decompose Fourier modes in the helicity basis (i.e. into a scalar and a vector part) $q^i = \delta^{ij}q_j = q_{(+)}\bar{e}^i_{(+)} + q_{(-)}\bar{e}^i_{(-)} + q_{(0)}\bar{e}^i_{(0)}$ and $q_{(h)} = q_i\bar{e}^{*,i}_{(h)}$. For second order perturbations we choose to align the direction $h=0$ with the total Fourier mode $\bar{e}^i_{(0)} = k^i$. For perturbed vector quantities S, we make the identification $\bar{e}^a_{(h)} = \bar{e}^i_{(h)}$ in order to expand $S^a = S_{(+)}e^a_{(+)} + S_{(-)}e^a_{(-)} + S_{(0)}e^a_{(0)}$ with $S_{(h)} = S_a\bar{e}^{*,a}_{(h)}$.

In order to make the structure of equations compact, we introduce the notation

$$\mathcal{K}\{f_1,f_2\}(\vec{k}) \int \frac{d^3k_1 d^3k_2}{(2\pi)^{\frac{3}{2}}} \delta^3(\vec{k_1}+\vec{k_2}-\vec{k}) f_1(\vec{k_1}) f_2(\vec{k_2}) \tag{3.4}$$

that will be useful when dealing with term with product of first order quantities. Using the identities (related to the transformation under a rotation of the helicity basis) $i\epsilon^{abc}k_b e^{(\pm)}_b = \pm k e^a_{(\pm)}$ and $ie^{(\pm)*}_a \epsilon^{abc}k_b S_c = \pm k S_{(\pm)}$ and keeping track of the perturbation order we can rewrite the evolution equation for B^a as

$$(a^2 B^{(2)}_{(\pm)}(\vec{k}))' = \mp\, a^2 \left[E^{(2)}_{(\pm)} + \mathcal{K}\{(\Phi^{(1)} - \Psi^{(1)}), E^{(1)}_{(\pm)}\}(\vec{k})\right] \tag{3.5}$$

Rewriting this equation for the magnetogenesis explicitly using the expression for the generated electric field E^a we get

$$\begin{aligned}(a^2 B^{(2)}_{(\pm)}(\vec{k}))' =\mp\ & a^2 \frac{4\sigma_T\rho_\gamma}{3e}[V^{(2)}_{(\pm)} + \mathcal{K}\left\{\left(\frac{\delta\rho_\gamma}{\bar{\rho_\gamma}} + \Phi^{(1)} - \Psi^{(1)}\right), V^{(1)}_{(\pm)}\right\}(\vec{k}) - \\ & - \sum_h \mathcal{K}\left\{\frac{\kappa(\pm1,h)}{5}\Theta^{\pm1+h,(1)}, v^{(1)}_{B,(-h)}\right\}(\vec{k})] = \\ =\mp\ & a^2 \frac{4\sigma_T\rho_\gamma}{3e}\left[S^{(\pm)}_1(\vec{k}) + S^{(\pm)}_2(\vec{k}) + S^{(\pm)}_3(\vec{k})\right]\end{aligned}$$

where $V_{(h)} = v_{B,(h)} - v_{\gamma,(h)}$ and $\kappa(g,h) = \sqrt{(4-g^2)}\delta(h,0) - \sqrt{\frac{(2+g)(3+g)}{2}}\delta(h,+) - \sqrt{\frac{(2-g)(3-g)}{2}}\delta(h,-)$. Of these 3 sources $S_i(\vec{k})$ that will appear in the magnetic spectrum only the first one $S_1(\vec{k})$ is a second order contribution, whereas the other two $S_2(\vec{k})$ and $S_3(\vec{k})$ are product of two first order contributions.

In order to define the magnetic field power spectrum, we need to rely upon the power spectrum $P(k)$ of the initial gravitational potential $\Phi_{in}(\vec{k})$ in the deep radiation era, that is defined as

$$\left\langle \Phi_{in}(\vec{k})\Phi^*_{in}(\vec{q}) \right\rangle = \delta(\vec{k}-\vec{q})P(k) \tag{3.6}$$

The post-processing of the well-known primordial perturbations and their evolution is encoded in the so-called transfer function: the first and second order transfer functions for a variable X are defined (implicitly) as

$$X^{(1)}(\vec{k},\eta) = \chi^{(1)}(\vec{k},\eta)\Phi_{in}(\vec{k}) \tag{3.7}$$

$$X^{(2)}(\vec{k},\eta) = \mathcal{K}\{\chi^{(2)}(\vec{k_1},\vec{k_2},\eta)\Phi_{in}(\vec{k_1})\Phi_{in}(\vec{k_2})\}(\vec{k}) \tag{3.8}$$

where we require $\chi^{(2)}(\vec{k_1},\vec{k_2},\eta) = \chi^{(2)}(\vec{k_2},\vec{k_1},\eta)$ in our calculations. In our case from equation of magnetogenesis we can relate the transfer function of the magnetic field to the transfer functions of the sources S^a, noticing that a and $\bar{\rho_\gamma}$ depends on the conformal time η

$$a^2\mathcal{B}^{S_i}_{(\pm)}(\vec{k_1},\vec{k_2},\eta) = \frac{4\sigma_T k}{3e}\int^\eta d\eta' a^2(\eta')\bar{\rho_\gamma}(\eta')\mathcal{S}^{(\pm)}_i(\vec{k_1},\vec{k_2},\eta) \tag{3.9}$$

where the final time of integration should be chosen after the recombination.

3.2 The power spectrum and the estimate of the field strength

At this point our goal is to have an estimate of the quantity $\left\langle \vec{B}(\vec{x},\eta)\cdot\vec{B}^*(\vec{x}',\eta) \right\rangle$, where the bracket denotes an ensemble average, i.e. an average over many realizations of the stochastic magnetic field. Actually we can measure only one realization, but using the ergodic hyphothesis the spatial average over many independent patches of size $L \gg \frac{2\pi}{k}$ is a good approximation of the ensemble average, especially if the size L is larger than the cosmological horizon at the time when the magnetic field was generated. If the source terms are Gaussian random variables, using Wick's theorem we obtain in Fourier space with $\mathcal{B}_{(\pm)}(\vec{k_1},\vec{k_2},\eta) = \sum_i \mathcal{B}^{S_i}_{(\pm)}(\vec{k_1},\vec{k_2},\eta)$

$$\left\langle \vec{B}(\vec{k},\eta)\cdot\vec{B}^*(\vec{k}',\eta) \right\rangle = \delta^3(\vec{k}-\vec{k}')P_B(k,\eta) = \frac{\delta^3(\vec{k}-\vec{k}')}{(2\pi)^3}\sum_{h=\pm}\int d^3q P(q)P(|\vec{k}-\vec{q}|)\times \tag{3.10}$$

$$\left[|\mathcal{B}_{(h)}(\vec{q},\vec{k}-\vec{q},\eta)|^2 + \mathcal{B}_{(h)}(\vec{q},\vec{k}-\vec{q},\eta)\mathcal{B}^*_{(h)}(\vec{k}-\vec{q},\vec{q},\eta)\right] \tag{3.11}$$

where the $\delta^3(\vec{k}-\vec{k}')$ is a conseguence of spatial homogeneity. In the following we consider the quantity $k^3P_B(k,\eta)$, since it is dimensionless, and the conformal time of matter-radiation equality η_{eq}. We can estimate three different kind of contributions to $P_B(k,\eta)$ related to the three sources S_i on super Hubble scales (see figure 2):

- $\sqrt{k^3P_B(k,\eta)} \propto k^4\frac{\eta}{\eta_{eq}} \to$ Second order contribution velocity $\Delta v_{B\gamma}$ (dot-dashed in the plot): it can be estimated direcly from the tight coupling expansion of the evolution equation for the vorticity of baryons (in the tetrad basis up to second order)

- $\sqrt{k^3 P_B(k,\eta)} \propto k^4 \frac{\eta^2}{\eta_{eq}^2} \rightarrow$ Quadratic term in velocity and density $\delta v_{B\gamma} \frac{\delta\rho_\gamma}{\bar{\rho_\gamma}}$ (dashed in the plot): it can be estimated direcly from the tight coupling expansion of the velocity difference between baryons and photons relative to other perturbations like $\frac{\delta\rho_\gamma}{\bar{\rho_\gamma}}$
- $\sqrt{k^3 P_B(k,\eta)} \propto k^4 \frac{\eta}{\eta_{eq}} \rightarrow$ Quadratic term in velocity and anisotropic stress $v_B \Theta_2$ (dotted in the plot): it can be estimated direcly from the tight coupling expansion of the source $\Theta_2(k,\eta)$

Instead on small scales the behaviour of the power spectrum is more difficult to understand, and with a linear reasonable approximation can be $\sqrt{k^3 P_B(k,\eta)} \propto \sqrt{k}$. The largest contribution to the magnetic field comes from the last period of generation, after recombination: the decoupling of photons and baryons enhance the departure from the tight-coupling regime via non adiabatic pressure perturbations, which source the total vorticity[3]. This is due mainly to the residual ionized fraction that is still non-zero after the recombination, even if the counterbalancing effect of the redshifting background energy density of photons will slow and then stop the generation of the magnetic field at later times [13].

Using a Gaussian window function $W_G(y) = \frac{1}{V_G} e^{-\frac{y^2}{2\lambda^2}}$ for a comoving scale λ, where $V_G = \int d^3y e^{-\frac{y^2}{2\lambda^2}} = \lambda^3 (2\pi)^{\frac{3}{2}}$ is the normalization volume, we can also obtain the magnetic field amplitude

$$B_\lambda^2 = \frac{1}{V_G} \int d^3y \left\langle \vec{B}(\vec{x}) \cdot \vec{B}^*(\vec{x}+\vec{y}) \right\rangle e^{-\frac{y^2}{2\lambda^2}} = \frac{1}{2\pi^2} \int_0^{+\infty} k^2 P_B(k) e^{-\frac{k^2\lambda^2}{2}} \tag{3.12}$$

The order of magnitude of the strength of the magnetic field is around $10^{-30} \div 10^{-29} G$, on comoving scales of the order of $1 \div 10$ Mpc (see figure 3), that is too low for the typical seed magnetic field required by dynamo-like mechanism.

[3] The vorticity of the fluid of baryons and photons do not combine in a linear way

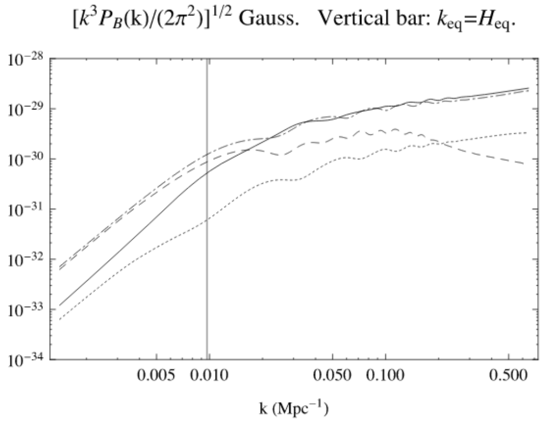

Figure 2: Dimensionless power spectrum of the magnetic field, with global (solid line) and single source contributions

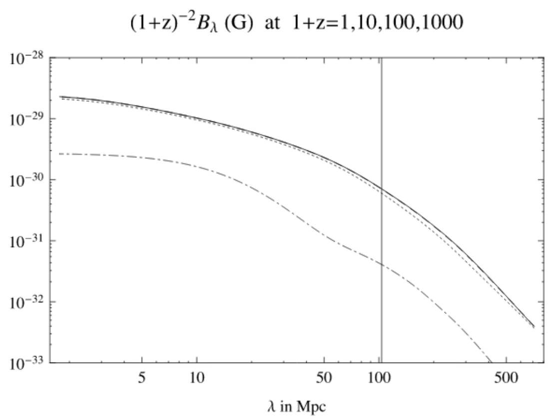

Figure 3: Comoving magnetic field strength at different redshift $z = 1, 10, 100, 1000$, represented by a solid, dashed, dotted and dot-dashed lines respectively

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