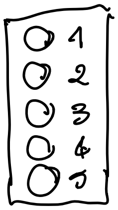
Ex. Consider a brand lock with 5 buttons. We need to use all buttons, and a combination like 12-35-6 means that we need to push 13, 35 and 6 (without reusing buttons). How many possible combinations are there, given a certain number of buttons and pushes?



Sol. Represent 12-35-6 as an ordered set partition $\{\{1,2\},\{2,5\},\{6\}\}$. Letting

- n = number of buttons
- K = groups of button pushes

we get $\underbrace{S(n,K)}_{\text{ways to split buttons into } K \text{ blocks}} \cdot \underbrace{K!}_{\text{ways to order the blocks}}$ combinations

Bonus ex. Represent the combination as a surjective function, and do again the counting!

Ex. How many set partitions of $[10]$ have type $(3,2,2,1,1,1)$?

Sol. Split 10 into sets A,B,C,D,E,F of sizes $3,2,2,1,1$ in $\Rightarrow \binom{10}{3,2,2,1,1,1}$ ways

Then, taking into account that we overcount the reorderings of (B,C) and (D,E,F):

Result: $\dfrac{\binom{10}{3,2,2,1,1,1}}{1!\,2!\,3!} = \dfrac{151200}{12} = 12600$ ← Theorem 5.22

Ex. 5.27 Find a closed formula for $S(n,n-2)$ if $n \geq 2$.

Sol. Start with $S(n,n)$: in how many ways can we split $[n]$ distinct objects into n identical non-empty boxes? 1

For $S(n,n-1)$ $\quad \underbrace{\boxed{1}\ \boxed{1}\ \boxed{1}\ - - -\ \boxed{1}\ \boxed{1}}_{n-2}\ \boxed{2}$

and therefore $S(n,n-1)$ = number of ways of choosing two elements from $[n] = \binom{n}{2} = \frac{n(n-1)}{2}$

For $S(n,n-2)$

$A: \underbrace{\boxed{1}\ \boxed{1}\ \boxed{1}\ - - - \ \boxed{1}\ \boxed{1}}_{n-3}\ \boxed{3}$ $\qquad \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

$B: \underbrace{\boxed{1}\ \boxed{1}\ \boxed{1}\ - - - \ \boxed{1}}_{n-4}\ \boxed{2}\ \boxed{2}$ $\qquad \binom{n}{2}\binom{n-2}{2}\frac{1}{2} =$

$$= \frac{1}{2}\,\frac{n(n-1)}{2}\,\frac{(n-2)(n-3)}{2}$$

Result: $A+B = \frac{n(n-1)(n-2)(3n-5)}{24}$

Ex. (similar to 6.38, 6.39) How many permutations $p \in S_6$ satisfy $p^6 = 1$

Sol. Only cycles of length 1, 2, 3, 6 give the identity once raised to the sixth power

Method 1: We count permutations where all cycles are of specified length:

$(a,b,c) \to$ length of cycles $(1,2,3)$ $\qquad$ Theorem 6.9

(6,0,0)	(0,3,0)	(0,0,2)	(3,0,1)	(4,1,0)	(2,2,0)	(0,1,1)
$\frac{6!}{6!\,1^6}$	$\frac{6!}{3!\,2^3}$	$\frac{6!}{2!\,3^2}$	$\frac{6!}{3!\,1!\,1^3 3^1}$	$\frac{6!}{4!\,1!\,1^4 2^1}$	$\frac{6!}{2!\,2!\,2^1 3^1}$	$\frac{6!}{1!\,1!\,2^2 3^1}$

6 1-cycles $\qquad$ 2 3-cycles

1 6-cycle

+

(0,0,0,0,0,1)
$\frac{6!}{1!\,6^1}$

Alternative: subtract configurations that contain a forbidden cycle length (4 or 5)

$$6! - \quad \overset{(2,0,0,1)}{\frac{6!}{2!\,1!\,1^2\,4^1}} \;\cdot\; \overset{(0,1,0,1)}{\frac{6!}{1!\,1!\,2^1\,4^1}} \quad \overset{(1,0,0,0,1)}{\frac{6!}{1!\,1!\,1^1\,5^1}}$$

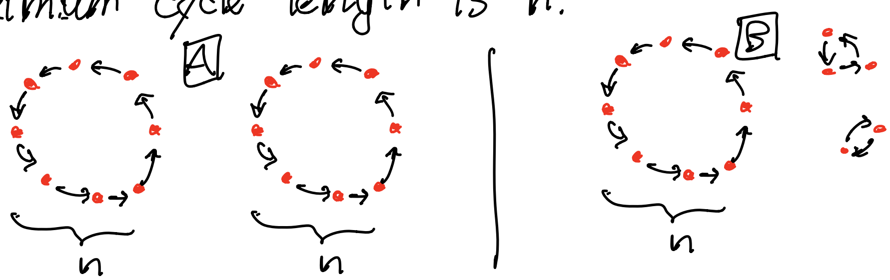
Result: 396 (= 720 − 324)

Ex. 6.31

What is the number of (2n)-permutations whose longest cycle is of length n?

Sol. We need to count the number of permutations $p \in S_{2n}$ such that in the disjoint cycle representation of p the maximum cycle length is n.

Two cases:



$\boxed{A}$ $\displaystyle\binom{2n}{n} \times \left[(n-1)!\right]^2 \times \frac{1}{2}$

- $\binom{2n}{n}$: n-elemens subsets of {2n} used for the first cycle
- $[(n-1)!]^2$: cyclic orderings of the elements of the two n-cycles
- $\frac{1}{2}$: double counting for choosing the first cycle

$\boxed{B}$ $\displaystyle\binom{2n}{n} \times (n-1)! \times \left[n! - (n-1)!\right]$

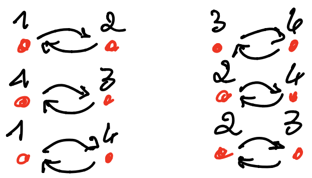
- $\binom{2n}{n}$: n-elemens subsets of {2n} used for the n-cycle
- $(n-1)!$: cyclic orderings of the elements of the n-cycle
- $n! - (n-1)!$: number of permutations which leave invariant the leftover elements (excluding those who would form another n-cycle $\Rightarrow \boxed{A}$)

Result: $\boxed{A} + \boxed{B} = (2n-1)!\left(\frac{2n-1}{n}\right)$

Check $n=1 \Rightarrow 1$

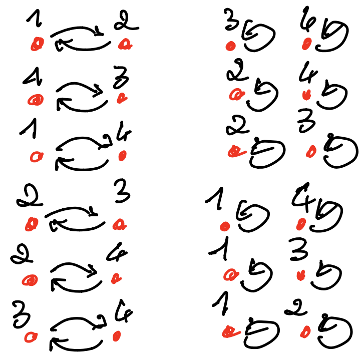
$n=2 \Rightarrow 3!\left(\frac{3}{2}\right) = 9$

$\boxed{A}$



$= 3$

$\boxed{B}$



$= 6$

Remark (HW 1 Assignment)

Geometric series: $\sum_{k=0}^{n-1} a^k = \left(\frac{1-a^n}{1-a}\right)$